Supplementary Material for “Stationary subspace analysis of
nonstationary covariance processes: eigenstructure description and
testing”

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A Additional figures for Section 6

A.1 Figures for Section 6.1

Figure 1: Model 1: Histograms of the estimates of $d$ for the indicated sample sizes for the two competing methods: DSSA (Sundararajan and Pourahmadi (2018)) and VC (proposed method). The true value is $d = 1$.

Figure 2: Model 2: Histograms of the estimates of $d$ for the indicated sample sizes for the two competing methods: DSSA and VC (proposed method). The true value is $d = 2$. 
Figure 3: Model 3: Histograms of the estimates of $d$ for the indicated sample sizes for the two competing methods: DSSA and VC (proposed method). The true value is $d = 2$.

Figure 4: Model 4: Histograms of the estimates of $d$ for the indicated sample sizes for the two competing methods: DSSA and VC (proposed method). The true value is $d = 3$. The value of $\rho$ is set to 0.5.

A.2 Figures for Section 6.2

Figure 5: Model 2 - Top: Plot of $D_1(\hat{B}_1(u))$ against $u$ for the competing methods DSSA and VC and several sample sizes. VC (avg.) in triangles in squares and DSSA in solid circles. Bottom: Analogous plot but with measure $D_2(\hat{B}_1(u))$ against $u$. 
Figure 6: Model 3 - Top: Plot of $D_1(\hat{B}_1(u))$ against $u$ for the competing methods DSSA and VC and several sample sizes. VC (avg.) in triangles, VC (min.) in squares and DSSA in solid circles. Bottom: Analogous plot but with measure $D_2(\hat{B}_1(u))$ against $u$.

Figure 7: Model 4 - Top: Plot of $D_1(\hat{B}_1(u))$ against $u$ for the competing methods DSSA and VC and several sample sizes. VC (avg.) in triangles, VC (min.) in squares and DSSA in solid circles. Bottom: Analogous plot but with measure $D_2(\hat{B}_1(u))$ against $u$. 

B Additional figures and tables for Section 7

Figure 8: Histogram of the dimension estimates \(d\) by the two competing methods based on the 144 trials.

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<th>S1</th>
<th>S2</th>
<th>S3</th>
<th>S4</th>
<th>S5</th>
<th>S6</th>
<th>S7</th>
<th>S8</th>
<th>S9</th>
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Table 1: Out-of-sample classification accuracy (in %) for the 9 subjects S1-S9 for the two indicated methods.

Figure 9: \(p = 22\). Histogram of the stationary subspace dimension estimates based for VC method based on the 144 trials.
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<th>$d$</th>
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Table 2: Out-of-sample classification accuracy (in %) for the 9 subjects S1-S9 corresponding to $d = 7, 9, 11, \text{ and } 13$ for the VC and DSSA methods with $p = 22$.

C Consistent estimator of fourth moment

The asymptotic results in Propositions 4.1 and 4.2 involve the quantity $\mu_4 = \mathbb{E}(Y_{i,t}^2 - 1)^2 = \mathbb{E}Y_{i,t}^4 - 1$, that is, the fourth moment of the variables $Y_{i,t}$ in the VC model (1.2). One natural estimator of $\mu_4$ is to set

$$\hat{\mu}_4 = \frac{1}{Tp} \sum_{i=1}^{p} \sum_{t=1}^{T} \left( (\hat{A}(\frac{t}{T})^{-1}X_t)_i^4 - 1 \right).$$  \hspace{1cm} (C.1)

But analyzing this estimator would require being able to control $\hat{A}(u)$ uniformly across $u$. This is certainly possible, and is essentially done in connection to global tests, but would require more stringent assumptions than those used for local tests in Appendix A.1 of the paper. Instead of (C.1), a consistent estimator of $\mu_4$ can be constructed under the assumptions of Appendix A.1 of the paper based on the following argument.

To get to the fourth moment of $Y_{i,t}$, we shall consider $(\text{tr}\{X_tX_t'\})^2$. Note that

$$\text{tr}\{X_tX_t'\} = \text{tr}\{A(\frac{t}{T})Y_tY_t'\} = \text{tr}\{A(\frac{t}{T})'A(\frac{t}{T})Y_tY_t'\} = \text{tr}\{A(\frac{t}{T})Y_tY_t'\}
= \text{tr}\{A^2(\frac{t}{T})(Y_tY_t' - I_p)\} + \text{tr}\{A^2(\frac{t}{T})\},$$
where we used the symmetry of $A(u)$ to write $A(\cdot)'A(\cdot) = A(\cdot)A(\cdot)' = A^2(\cdot)$. Then,

$$
\frac{1}{T} \sum_{t=1}^{T} (\text{tr}\{X_tX'_t\})^2 = \frac{1}{T} \sum_{t=1}^{T} \left(\text{tr}\{A^2(\frac{t}{T})(Y_tY'_t - I_p)\}\right)^2
+ \frac{2}{T} \sum_{t=1}^{T} \text{tr}\{A^2(\frac{t}{T})(Y_tY'_t - I_p)\} \text{tr}\{A^2(\frac{t}{T})\}
+ \frac{1}{T} \sum_{t=1}^{T} \left(\text{tr}\{A^2(\frac{t}{T})\}\right)^2 =: R_1 + R_2 + R_3. \tag{C.2}
$$

Under the assumptions of Appendix A.1 of the paper, note that $R_2 \to 0$ a.s. Indeed, this follows from the following general argument that will be used on several occasions below. After expanding the traces, an entry in $R_2$ can be expressed as

$$
\frac{1}{T} \sum_{t=1}^{T} b(\frac{t}{T})Z_t, \tag{C.3}
$$

where $b(\cdot)$ is continuously differentiable and $Z_t$’s are i.i.d. with zero mean. By the summation by parts formula,

$$
\frac{1}{T} \sum_{t=1}^{T} b(\frac{t}{T})Z_t = \frac{1}{T} \sum_{t=1}^{T-1} \left( \sum_{s=1}^{t} Z_s \right) \left( b(\frac{t}{T}) - b(\frac{t+1}{T}) \right) + \frac{b(1)}{T} \sum_{t=1}^{T} Z_t - \frac{b(1/T)}{T} \sum_{t=1}^{T-1} Z_t.
$$

All three last terms converge to 0 a.s. by the law of large numbers. For the first term, in particular, this follows from bounding it by

$$
C \sum_{s=1}^{T} \left| \sum_{t=1}^{T} Z_s \right| \frac{t}{T^2} = C \sum_{s=1}^{T} \left| \frac{1}{t} \sum_{s=1}^{t} Z_s \right| \frac{t}{T^2}
$$

and noting that $\frac{1}{T} \sum_{s=1}^{t} Z_s \to 0$ a.s. as $t \to \infty$.

Under the assumptions of Appendix A.1 of the paper, $R_3 \to \int_0^1 (\text{tr}\{A^2(u)\})^2 du$. For the term $R_1$, note that

$$
R_1 = \frac{1}{T} \sum_{t=1}^{T} \left( \sum_{i,j=1}^{p} A^{2}_{ij}(\frac{t}{T})(Y_{t,i}Y'_{t,j} - \delta_{ij}) \right)^2 =: \frac{1}{T} \sum_{t=1}^{T} R_{1,t},
$$

where $\delta_{ij} = 1$ if $i = j$, and = 0 otherwise. By separating the sum $R_{1,t}$ into that over $i = j$ and that over $i < j$, and taking the square, we can further write

$$
R_{1,t} = R_{1,1,t} + R_{1,2,t} + R_{1,3,t} + R_{1,4,t} + R_{1,5,t},
$$
where

\[ R_{1,1,t} = \sum_{i=1}^{p} (A_{ii}^2(t/T))^2 (Y_{i,t}^2 - 1)^2, \]

\[ R_{1,2,t} = \sum_{i\neq i'} A_{ii'}^2(t/T) A_{ii'}^2(t/T) (Y_{i'}^2 - 1)(Y_{i',t}^2 - 1), \]

\[ R_{1,3,t} = 4 \sum_{i'=1}^{p} A_{ii'}^2(t/T) (Y_{i'}^2 - 1) \sum_{i<j} A_{ij}^2(t/T) Y_{i,t} Y_{j,t}, \]

\[ R_{1,4,t} = 4 \sum_{i<j} (A_{ij}^2(t/T))^2 Y_{i,t}^2 Y_{j,t}, \]

\[ R_{1,4,t} = 4 \sum_{i<j} \sum_{i' < j'} 1_{\{i \neq i' \text{ or } j \neq j'\}} A_{ij}^2(t/T) A_{ij'}^2(t/T) Y_{i,t} Y_{j,t} Y_{i',t} Y_{j',t}. \]

By the same reasoning following (C.3), for \( k = 2, 3, 5 \)

\[ \frac{1}{T} \sum_{t=1}^{T} R_{1,k,t} \rightarrow 0 \quad \text{a.s.} \]

and

\[ \frac{1}{T} \sum_{t=1}^{T} R_{1,1,t} \rightarrow \mu_4 \int_0^1 \sum_{i=1}^{p} (A_{ii}^2(u))^2 du, \quad \frac{1}{T} \sum_{t=1}^{T} R_{1,4,t} \rightarrow 4 \int_0^1 \sum_{i<j} (A_{ij}^2(u))^2 du \quad \text{a.s.} \]

By gathering the above observations, it follows from (C.2) that, almost surely,

\[ \frac{1}{T} \sum_{t=1}^{T} (\text{tr}\{X_t X_t'\})^2 \rightarrow \mu_4 \int_0^1 \sum_{i=1}^{p} (A_{ii}^2(u))^2 du + 4 \int_0^1 \sum_{i<j} (A_{ij}^2(u))^2 du =: \mu_4 I_1 + 4I_2. \quad (C.4) \]

We indicate how the integrals \( I_1 \) and \( I_2 \) can be estimated consistently, from which a consistent estimator of \( \mu_4 \) will follow.

The integrals \( I_1 \) and \( I_2 \) can be estimated through the following weighted \( U \)-statistics:

\[ \hat{I}_1 = \frac{1}{T(T-1)} \sum_{t_1 \neq t_2} \sum_{i=1}^{p} (X_{t_1} X_{t_1'} K_h(t_1/T - t_2/T)), \]

\[ \hat{I}_2 = \frac{1}{T(T-1)} \sum_{t_1 \neq t_2} \sum_{i<j} (X_{t_1} X_{t_1'} K_h(t_1/T - t_2/T)). \]

The convergence of \( \hat{I}_1 \) and \( \hat{I}_2 \) to \( I_1 \) and \( I_2 \), respectively, can be attained by using Proposition A.3 of the paper, which also yields convergence rates.

References