Technical appendix for "Local and global rank tests for multivariate varying-coefficient models"

Stephen G. Donald
University of Texas at Austin

Natércia Fortuna
CEF.UP, Universidade do Porto

Vladas Pipiras
University of North Carolina at Chapel Hill

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We gather here a number of proofs that were omitted from the article "Local and global rank tests for multivariate varying-coefficient models". The notation follows that of the article. References to the article will be made by adding square brackets, for example, Theorem [3.1] will refer to Theorem 3.1 in the article.

The proof of Theorem [3.1] can be found in Section 1 below. The proofs for Appendix [A] are in Section 2 below. Section 3 contains a number of auxiliary results used in this technical appendix and the article itself.

1 Proof of Theorem [3.1]

It is enough to prove the theorem for $d_0 = 0$, that is, $\Theta(z) = \theta(z)$, $\Psi(z) = \psi(z)$, $\Omega(z) = w(z) = (\psi(z)^{-1} \otimes \Sigma)\|K\|_2^2$ and similar expressions with the hats. Observe that

$$\widehat{\theta}(z) = \theta(z) + (\widehat{\Delta}_1(z) + \widehat{\Delta}_2(z))\ \widehat{\psi}(z)^{-1}, \tag{1.1}$$

where

$$\widehat{\Delta}_1(z) = \frac{1}{N}\sum_{i=1}^{N}(\theta(Z_i) - \theta(z))v(X_i)v(X_i)'K_h(z - Z_i),\ \widehat{\Delta}_2(z) = \frac{1}{N}\sum_{i=1}^{N} U_i v(X_i)'K_h(z - Z_i).$$

To prove the theorem, it is enough to show that

$$\widehat{\psi}(z) \ \stackrel{p}{\longrightarrow}\ \psi(z) \tag{1.2}$$

$$\begin{aligned}\widehat{\Delta}_1(z) &= o_p\Big((Nh^q)^{-1/2}\Big), &(1.3)\\ (Nh^q)^{1/2}\widehat{\Delta}_2(z) &\xrightarrow{d} \mathcal{N}(0, w_0(z)), &(1.4)\\ \widehat{\Sigma} &\xrightarrow{p} \Sigma, &(1.5)\end{aligned}$$

where $w_0(z) = (\psi(z) \otimes \Sigma)\|K\|_2^2$. The convergence (1.5) follows from Proposition 3.5 below.

The convergence (1.2) is standard. Letting $M^2 = MM'$ for a matrix M, consider

$$E(\widehat{\psi}(z) - \psi(z))^2 = E\widehat{\psi}(z)^2 - E\widehat{\psi}(z)\psi(z)' - \psi(z)E\widehat{\psi}(z)' + \psi(z)^2.$$

Since $E\widehat{\psi}(z) = Ev(X_i)v(X_i)'K_h(z - Z_i) = E\phi(Z_i)K_h(z - Z_i) = \int_{\mathcal{H}_z} \phi(z_i)p(z_i)K_h(z - Z_i)dz_i$, by applying Proposition 3.1, (i), and using the assumptions on $\phi(z)$ and $p(z)$, we obtain that $E\widehat{\psi}(z) = \phi(z)p(z) + O(h^s) = \psi(z) + O(h^s)$. As for $E\widehat{\psi}(z)^2$, by using independence of (X_i, Z_i) and (X_j, Z_j) for $i \neq j$, we have

$$\begin{aligned}E\widehat{\psi}(z)^2 &= \frac{\|K\|_2^2}{Nh^q} E\left((v(X_i)v(X_i)')^2 K_{2,h}(z - Z_i)\right) + \frac{N-1}{N}\Big(Ev(X_i)v(X_i)'K_h(z - Z_i)\Big)^2\\ &= \frac{\|K\|_2^2}{Nh^q} E\phi_2(Z_i)K_{2,h}(z - Z_i) + \frac{N-1}{N}\Big(E\phi(Z_i)K_h(z - Z_i)\Big)^2.\end{aligned}$$

By using Proposition 3.1, (i), the first term above is of the order $O((Nh^q)^{-1})$. The order of the second term is that of $(E\widehat{\psi}(z))^2 = \psi(z)^2 + O(h^s)$. Combining all asymptotic relations above yields $\widehat{\psi}(z) = \psi(z) + O_p(h^s + (Nh^q)^{-1/2})$.

Similarly as above,

$$\begin{aligned}E(\widehat{\Delta}_1(z))^2 &= \frac{\|K\|_2^2}{Nh^q} E\Big((\theta(Z_i) - \theta(z))\phi_2(Z_i)(\theta(Z_i) - \theta(z))'K_{2,h}(z - Z_i)\Big)\\ &\qquad + \frac{N-1}{N}\Big(E(\theta(Z_i) - \theta(z))\phi(Z_i)K_h(z - Z_i)\Big)^2\\ &= o\Big((Nh^q)^{-1}\Big) + O(h^{2s}) = o((Nh^q)^{-1}).\end{aligned}$$

To show (1.4), write $(Nh^q)^{1/2}\text{vec}(\widehat{\Delta}_2(z)) = N^{-1/2}\sum_{i=1}^N \xi_{N,i}$ with $\xi_{N,i} = h^{q/2}(v(X_i) \otimes I_m)U_iK_h(z - Z_i)$. By the Cramér-Wold theorem, we need to show that

$$\frac{1}{\sqrt{N}}\sum_{i=1}^N \eta_{N,i} \xrightarrow{d} \mathcal{N}(0, \omega_1(z)), \qquad (1.6)$$

where $\eta_{N,i} = \lambda'\xi_{N,i}$, $\lambda \in \mathbb{R}^{mn} \setminus \{0\}$ is an arbitrary vector and $\omega_1(z) = \lambda'\omega_0(z)\lambda$. By using the Lyapunov's version of Central Limit Theorem for triangular arrays and since $E\eta_{N,i} = 0$, this follows from

$$\frac{E\eta_{N,1}^4}{N(E\eta_{N,1}^2)^2} \to 0, \tag{1.7}$$

$$E(\eta_{N,1})^2 \to \omega_1(z). \tag{1.8}$$

The convergence (1.8) follows from $E\xi_{N,1}^2 \to w_0(z)$. For this, observe that $E\xi_{N,1}^2 = \|K\|_2^2 E((v(X_i)\otimes I_m)U_i)^2 K_{2,h}(z-Z_i) = \|K\|_2^2 E(\phi(Z_i)\otimes\Sigma)K_{2,h}(z-Z_i) = \omega_0(z) + O(h^2)$. For the convergence (1.7), observe that $E\eta_{N,1}^4 = \|K\|_4^4 h^{-q} E(\lambda'(v(X_i)\otimes I_m)U_i\lambda)^4 K_{4,h}(z-Z_i) = O(h^{-q})$ by using Proposition 3.1, (i). By using (1.8), we deduce the convergence (1.7).

2 Proofs for Appendix [A]

We shall write

$$\widehat{\xi} = O_{p,\sup}(a_N)$$

for $\widehat{\xi} = \widehat{\xi}(z)$ when $a_N^{-1}\sup|\widehat{\xi}(z)| = O_p(1)$.

Proof of Lemma [A.1]: Let $T = Nh^q$. As on p. 173 of Robin and Smith (2000), $\widehat{\lambda}_i$ satisfy

$$0 = \det\Big(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma}\Big) = \det\Big((\widetilde{C}\ \ T^{1/2}C)'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})(\widetilde{C}\ \ T^{1/2}C)\Big)$$
$$= \det\begin{pmatrix} \widetilde{C}'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})\widetilde{C} & \widetilde{C}'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})T^{1/2}C \\ T^{1/2}C'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})\widetilde{C} & TC'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})C \end{pmatrix}.$$

By using the relation $\det((A\ C; B\ D)) = \det(A)\det(D - BA^{-1}C)$, we further obtain that

$$0 = \det(\widehat{S})\det\Big(\widehat{W} - T\widehat{\lambda}_i\widehat{V}^{-1}\Big), \tag{2.1}$$

where $\widehat{S} = \widetilde{C}'(\widehat{g}\widehat{\psi}^{-1}\widehat{g}' - \widehat{\lambda}_i\widehat{\Sigma})\widetilde{C}$,

$$\widehat{V}^{-1} = C'\widehat{\Sigma}C + \widehat{\lambda}_i C'\widehat{\Sigma}\widetilde{C}\,\widehat{S}^{-1}\,\widetilde{C}'\widehat{\Sigma}C - C'\widehat{\Sigma}\widetilde{C}\,\widehat{S}^{-1}\,\widetilde{C}'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C - C'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'\widetilde{C}\,\widehat{S}^{-1}\,\widetilde{C}'\widehat{\Sigma}C$$

and

$$\widehat{W} = TC'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C - TC'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'\widetilde{C}\,\widehat{S}^{-1}\,\widetilde{C}'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C. \tag{2.2}$$

By using Propositions 3.4–3.6 and the smoothness of $\widetilde{C}$ by Proposition 3.3, observe that

$$\widehat{S} = \widetilde{C}'g\psi^{-1}g'\widetilde{C} + O_{p,\sup}\left((Nh^q/\ln N)^{-1/2}\right). \tag{2.3}$$

As in the proof of Lemma A.1 of Robin and Smith (2000), observe also that

$$\widetilde{C}'g = \mathrm{diag}(\lambda_m^{1/2},\ldots,\lambda_{m-l+1}^{1/2})\,\widetilde{D}_*', \tag{2.4}$$

where $D_0^{-1} = (\widetilde{D}_* \; D_*)'$ is the inverse of D_0 with a $n\times(n-l)$ matrix $\widetilde{D}_*$. Since $\widetilde{D}_*'\psi^{-1}\widetilde{D}_* = I_{n-l}$ by using $D_0'\psi D_0 = I_n$, we obtain from (2.3) and (2.4) that

$$\widehat{S} = \operatorname{diag}(\lambda_m, \ldots, \lambda_{m-l+1}) + O_{p,\sup}\left((Nh^q/\ln N)^{-1/2}\right). \tag{2.5}$$

Relation (2.5) shows that, asymptotically, $\det(\widehat{S}) > 0$. Hence, in view of (2.1), we may suppose without loss of generality that $\det(\widehat{W} - T\widehat{\lambda}_i\widehat{V}^{-1}) = 0$, that is, $T\widehat{\lambda}_i$ are the eigenvalues of the matrix $\widehat{W}\widehat{V}$. The matrix $\widehat{V}$ is symmetric and its eigenvalues are positive asymptotically because $\widehat{V} \to_p C'\Sigma C$. Therefore, we may suppose that $\widehat{V}$ is positive definite, and that $T\widehat{\lambda}_i$ are the eigenvalues of $\widehat{V}^{1/2}\widehat{W}\widehat{V}^{1/2}$. Since this matrix is symmetric, applying the Wielandt-Hoffman theorem (Golub and Van Loan (1996), Stewart and Sun (1990)), we obtain that

$$\sup|T\widehat{\lambda}_i - T\widehat{\eta}_i|^2 \leq \sup\sum_{i=1}^{m-l}|T\widehat{\lambda}_i - T\widehat{\eta}_i|^2 \leq \sup\left|\widehat{V}^{1/2}\widehat{W}\widehat{V}^{1/2} - TC'(\widehat{g}-g)DD'(\widehat{g}-g)'C\right|^2. \tag{2.6}$$

Finally, we bound the right-hand side of (2.6) by examining the terms of the matrix $\widehat{W}$ in (2.2). By using $C'g = 0$ and Proposition 3.4, we have

$$TC'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C = TC'(\widehat{g}-g)\psi^{-1}(\widehat{g}-g)'C + O_{p,\sup}\left((Nh^q/\ln^3 N)^{-1/2}\right). \tag{2.7}$$

Similarly, by using (2.4), (2.5) and the relation $\psi^{-1}\widetilde{D}_* = \widetilde{D}$, we obtain that

$$TC'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'\widetilde{C}\widehat{S}^{-1}\widetilde{C}'\widehat{g}\widehat{\psi}^{-1}\widehat{g}'C = TC'(\widehat{g}-g)\widetilde{D}\widetilde{D}'(\widehat{g}-g)'C + O_{p,\sup}\left((Nh^q/\ln^3 N)^{-1/2}\right). \tag{2.8}$$

By using $\psi^{-1} - \widetilde{D}\widetilde{D}' = DD'$, we conclude from (2.7) and (2.8) that

$$\widehat{W} = TC'(\widehat{g}-g)DD'(\widehat{g}-g)'C + O_{p,\sup}\left((Nh^q/\ln^3 N)^{-1/2}\right). \tag{2.9}$$

By using Proposition 3.5 and the fact $C'\Sigma C = I_{m-l}$, we have $\widehat{V}^{1/2} = I_{m-l} + O_{p,\sup}((Nh^q/\ln N)^{-1/2})$. Hence, in view of (2.9), the right-hand side of (2.6) is $O_{p,\sup}((Nh^q/\ln^3 N)^{-1})$. This implies the desired result. □

Proof of Lemma [A.2]: Applying the Poincaré separation theorem (Magnus and Neudecker (1999), p. 209, or Rao (1973), p. 65), we have $\widehat{\eta}_i \leq \widehat{\zeta}_i$, where $\widehat{\zeta}_i$ are the ordered eigenvalues of the matrix $C_1'(\widehat{g}-g)DD'(\widehat{g}-g)'C_1$. These are also the eigenvalues of the matrix $D'(\widehat{g}-g)'C_1C_1'(\widehat{g}-g)D$. Applying the Poincaré separation theorem again, we further obtain that $\widehat{\eta}_i \leq \widehat{\xi}_i$, where $\widehat{\xi}_i$ are the eigenvalues of $D_1'(\widehat{g}-g)'C_1C_1'(\widehat{g}-g)D_1$. These are also the eigenvalues of the matrix $C_1'\widehat{g}D_1D_1'\widehat{g}'C_1$. $\square$

Proof of Lemma [A.3]: Write $\widehat{S}_{1,glb}(r) = \|K\|_2^2 N^{-1}\sum_{i=1}^N \beta_i$, where

$$\beta_i = \int_{\mathcal{H}} \mathrm{tr}\{D_1'v(X_i)v(X_i)'D_1\}\mathrm{tr}\{C_1'U_iU_i'C_1\}K_{2,h}(z-Z_i)dz.$$

Observe that, since $E\mathrm{tr}\{C_1'U_iU_i'C_1\} = \mathrm{tr}\{C_1'\Sigma C_1\} = \mathrm{tr}\{I_{m-r}\} = m-r$ and $\mathrm{tr}\{D_1'\psi D_1\} = n-r$ using $D_1'\psi D_1 = I_{n-r}$, we have

$$\begin{aligned} E\beta_i &= (m-r)\int_{\mathcal{H}} E\mathrm{tr}\{D_1'\phi(Z_i)D_1\}K_{2,h}(z-Z_i)dz \\ &= (m-r)\int_{\mathcal{H}}(\mathrm{tr}\{D_1'\psi D_1\}+O(h^2))dz = (m-r)(n-r)|\mathcal{H}|+O(h^2), \qquad (2.10) \end{aligned}$$

where we also used Proposition 3.1, (i), below and the assumption on ψ. Similarly, using the notation $A^2 = AA'$,

$$E\beta_i^2 = \int_{\mathcal{H}^2}\epsilon(z_1,z_2)E\phi_{2,D}(Z_i,z_1,z_2)K_{2,h}(z_1-Z_i)K_{2,h}(z_2-Z_i)dz_1dz_2,$$

where $\phi_{2,D}(z_i,z_1,z_2) = E(\mathrm{tr}\{(D_1(z_2)'v(X_i)v(X_i)'D_1(z_1))^2\}|Z_i = z_i)$, $\epsilon(z_1,z_2) = E\mathrm{tr}\{(C_1(z_2)'U_iU_i'C_1(z_1))^2\}$. By using Proposition 3.1, (ii) and (iii), and the assumption on ϕ_2, we further get that

$$\begin{aligned} E\beta_i^2 &= \int_{\mathcal{H}^2}\epsilon(z_1,z_2)(\phi_{2,D}(z_1,z_1,z_2)p(z_1)\overline{K}_{2,h}(z_1-z_2)+o(\overline{K}_{1,h}(z_1-z_2)))dz_1dz_2 \\ &= \int_{\mathcal{H}}\epsilon(z_1,z_1)\phi_{2,D}(z_1,z_1,z_1)p(z_1)dz_1+o(1). \qquad (2.11) \end{aligned}$$

The desired result follows from (2.10), (2.11) and $\widehat{S}_{1,glb}(r) = \|K\|_2^2E\beta_i + O_p((N^{-1}E\beta_i)^{1/2})$. $\square$

Proof of Lemma [A.4]: Write $\frac{\widehat{S}_{2,glb}(r)}{h^{q/2}} = \sum_{i<j}a_{ij} =: A_N$ as a second order U-statistic, where

$$a_{ij} = \frac{2h^{q/2}}{N}\int_{\mathcal{H}}\mathrm{tr}\{D_1'v(X_i)v(X_j)'D_1\}\mathrm{tr}\{C_1'U_iU_j'C_1\}K_h(z-Z_i)K_h(z-Z_j)dz.$$

By the Central Limit Theorem for U-statistics found in Proposition 3.2 of de Jong (1987), the desired results holds if (1) $\mathrm{Var}(A_N) \to \sigma^2 = 2|\mathcal{H}|\|\overline{K}\|_2^2(m-r)(n-r)$ and (2) $G_{N,i} = o(1)$ for $i = 1, 2$ and 4, where

$$G_{N,1} = \sum_{i<j}Ea_{ij}^4, \quad G_{N,2} = \sum_{i<j<k}(Ea_{ij}^2a_{ik}^2 + Ea_{ji}^2a_{jk}^2 + Ea_{ki}^2a_{kj}^2),$$

$$G_{N,4} = \sum_{i<j<k<l} (Ea_{ij}a_{ik}a_{lj}a_{lk} + Ea_{ij}a_{il}a_{kj}a_{kl} + Ea_{ik}a_{il}a_{jk}a_{jl}). \tag{2.12}$$

To show the part (1), observe that

$$\mathrm{Var}(A_N) = \frac{4h^q}{N^2}\frac{N(N-1)}{2}\int_{\mathcal{H}^2} E\mathrm{tr}\{C_1(z_2)'U_jU_j'C_1(z_1)C_1(z_1)'U_iU_i'C_1(z_2)\}\cdot$$

$$\cdot\, \mathrm{tr}\{D_1(z_2)'v(X_j)v(X_j)'D_1(z_1)D_1(z_1)'v(X_i)v(X_i)'D_1(z_2)\}\cdot$$

$$\cdot\, K_h(z_1 - Z_i)K_h(z_2 - Z_i)K_h(z_1 - Z_j)K_h(z_2 - Z_j)dz_1dz_2$$

$$= \frac{2h^q(N-1)}{N}\int_{\mathcal{H}^2} \epsilon_1(z_1, z_2)\mathrm{tr}\{(E\phi_{2,1}(Z_i, z_1, z_2)K_h(z_1 - Z_i)K_h(z_2 - Z_i))^2\}dz_1dz_2,$$

where $\epsilon_1(z_1, z_2) = \mathrm{tr}\{(C_1(z_2)'\Sigma C_1(z_1))^2\}$, $\phi_{2,1}(z_i, z_1, z_2) = D_1(z_2)'\phi_2(z_i)D_1(z_1)$ and we used the fact that $E(F_1(X_i)F_2(X_j)|Z_i = z_i, Z_j = z_j) = E(F_1(X_i)|Z_i = z_i)E(F_2(X_j)|Z_j = z_j)$. By using Proposition 3.1, (ii) and (iii), we further get

$$\mathrm{Var}(A_N) = \frac{2h^q(N-1)}{N}\int_{\mathcal{H}^2} \epsilon_1(z_1, z_2)\mathrm{tr}\{(\phi_{2,1}(z_1, z_1, z_2)p(z_1)\overline{K}_h(z_1 - z_2)$$

$$+ o(\overline{K}_{1,h}(z_1 - z_2)))^2\}dz_1dz_2$$

$$= 2\int_{\mathcal{H}} \epsilon_1(z_1, z_1)\mathrm{tr}\{(\phi_{2,1}(z_1, z_1, z_1)p(z_1))^2\}\|\overline{K}\|_2^2dz_1 + o(1)$$

$$= 2(m-r)(n-r)\|\overline{K}\|_2^2|\mathcal{H}| + o(1),$$

where we also used the facts $\epsilon_1(z_1, z_1) = \mathrm{tr}\{I_{m-r}\} = (m-r)$ and $\phi_{2,1}(z_1, z_1, z_1)p(z_1) = I_{n-r}$.

For the part (2), consider first $G_{N,1}$ in (2.12). Observe that

$$G_{N,1} = \frac{16h^{2q}}{N^4}\frac{N(N-1)}{2}\int_{\mathcal{H}^4} E\prod_{k=1}^{4} \mathrm{tr}\{D_1(z_k)'v(X_i)v(X_j)'D_1(z_k)\}\cdot$$

$$\cdot\, \mathrm{tr}\{C_1(z_k)'U_iU_j'C_1(z_k)\}K_h(z_k - Z_i)K_h(z_k - Z_j)dz_k.$$

By Proposition 3.1, (iii) and (iv) (see also Remark 3.1), $G_{N,1}$ behaves as (up to a constant)

$$\frac{h^{2q}}{N^2}E\mathrm{tr}\{D_1(Z_i)'v(X_i)v(X_j)'D_1(Z_i)\}^4\mathrm{tr}\{C_1(Z_i)'U_iU_j'C_1(Z_i)\}^4(\overline{K}_h(Z_i - Z_j))^4 1_{\{Z_i\in\mathcal{H}, Z_j\in\mathcal{H}\}}$$

$$= \frac{\|\overline{K}\|_4^4}{N^2h^q}E\mathrm{tr}\{D_1(Z_i)'v(X_i)v(X_j)'D_1(Z_i)\}^4\mathrm{tr}\{C_1(Z_i)'U_iU_j'C_1(Z_i)\}^4\overline{K}_{4,h}(Z_i - Z_j)1_{\{Z_i\in\mathcal{H}, Z_j\in\mathcal{H}\}}$$

$$= O\left(\frac{1}{N^2h^q}\right) = o(1),$$

since $Nh^q \to \infty$. Turning now to $G_{N,2}$ in (2.12), consider, for example, $G_{N,2,1} := \sum_{i<j<k} Ea_{ij}^2 a_{ik}^2$. Observe that

$$G_{N,2,1} = \frac{16h^{2q}}{N^4}\frac{N(N-1)(N-2)}{6}\int_{\mathcal{H}^4} EK_h(z_1-Z_i)K_h(z_2-Z_i)K_h(z_1-Z_j)K_h(z_2-Z_j)\cdot$$

$$\cdot\prod_{m=1}^{2}\mathrm{tr}\{D_1(z_m)'v(X_i)v(X_j)'D_1(z_m)\}\mathrm{tr}\{C_1(z_m)'U_iU_j'C_1(z_m)\}\cdot$$

$$\cdot\prod_{m=3}^{4}\mathrm{tr}\{D_1(z_m)'v(X_i)v(X_k)'D_1(z_m)\}\mathrm{tr}\{C_1(z_m)'U_iU_k'C_1(z_m)\}\cdot$$

$$\cdot K_h(z_3-Z_i)K_h(z_4-Z_i)K_h(z_3-Z_k)K_h(z_4-Z_k)dz_1dz_2dz_3dz_4.$$

Similarly to the case of $G_{N,1}$ above, $G_{N,2,1}$ behaves as (up to a constant)

$$\frac{h^{2q}}{N}E\mathrm{tr}\{D_1(Z_i)'v(X_i)v(X_j)'D_1(Z_i)\}^4\cdot$$

$$\cdot\,\mathrm{tr}\{C_1(Z_i)'U_iU_j'C_1(Z_i)\}^4(\overline{K}_h(Z_i-Z_j))^2(\overline{K}_h(Z_i-Z_k))^2 1_{\{Z_i\in\mathcal{H},Z_j\in\mathcal{H},Z_k\in\mathcal{H}\}}$$

$$= \frac{\|\overline{K}\|_2^4}{N}E\mathrm{tr}\{D_1(Z_i)'v(X_i)v(X_j)'D_1(Z_i)\}^4\mathrm{tr}\{C_1(Z_i)'U_iU_j'C_1(Z_i)\}^4\cdot$$

$$\cdot\,\overline{K}_{2,h}(Z_i-Z_j)\overline{K}_{2,h}(Z_i-Z_k)1_{\{Z_i\in\mathcal{H},Z_j\in\mathcal{H},Z_k\in\mathcal{H}\}} = O\left(\frac{1}{N}\right) = o(1).$$

Finally, for $G_{N,4}$ in (2.12), consider, for example, $G_{N,4,1} := \sum_{i<j<k<l} Ea_{ij}a_{ik}a_{lj}a_{lk}$. Observe that

$$G_{N,4,1} = \frac{16h^{2q}}{N^4}\frac{N(N-1)(N-2)(N-3)}{24}\int_{\mathcal{H}^4} F_4(Z_i,Z_j,Z_l,Z_k,z_1,z_2,z_3,z_4)\cdot$$

$$\cdot K_h(z_1-Z_i)K_h(z_2-Z_i)K_h(z_1-Z_j)K_h(z_3-Z_j)\cdot$$

$$\cdot K_h(z_3-Z_l)K_h(z_4-Z_l)K_h(z_2-Z_k)K_h(z_4-Z_k)dz_1dz_2dz_3dz_4,$$

where

$$F_4(z_i,z_j,z_l,z_k,z_1,z_2,z_3,z_4) = E\Big(\psi_{2,4}(z_1,X_i,X_j)\epsilon_4(z_1,U_i,U_j)\psi_{2,4}(z_2,X_i,X_k)\epsilon_4(z_2,U_i,U_k)\cdot$$

$$\cdot\,\psi_{2,4}(z_3,X_l,X_j)\epsilon_4(z_3,U_l,U_j)\psi_{2,4}(z_4,X_l,X_k)\epsilon_4(z_4,U_l,U_k)|Z_i=z_i,Z_j=z_j,Z_l=z_l,Z_k=z_k\Big)$$

with $\psi_{2,4}(z,x_1,x_2) = \mathrm{tr}\{D_1(z)'v(x_1)v(x_2)'D_1(z)\}$, $\epsilon_4(z,u_1,u_2) = \mathrm{tr}\{C_1(z)'u_1u_2'C_1(z)\}$. Using Proposition 3.1, (ii), one can argue that $G_{N,4,1}$ behaves as (up to a constant)

$$h^{2q}\int_{\mathcal{H}^4} F_4(z_1,z_1,z_4,z_4,z_1,z_2,z_3,z_4)p(z_1)^2p(z_4)^2\cdot$$

$$\cdot \overline{K}_h(z_1 - z_2)\overline{K}_h(z_1 - z_3)\overline{K}_h(z_3 - z_4)\overline{K}_h(z_2 - z_4)dz_1dz_2dz_3dz_4$$

and hence, by Proposition 3.1, (iv), as

$$h^{2q}\int_{\mathcal{H}^2} F_4(z_2, z_2, z_3, z_3, z_2, z_2, z_3, z_3)p(z_2)^2p(z_3)^2(\overline{\overline{K}}_h(z_2 - z_3))^2dz_2dz_3$$

$$= h^q\|\overline{\overline{K}}\|_2^2\int_{\mathcal{H}^2} F_4(z_2, z_2, z_3, z_3, z_2, z_2, z_3, z_3)\overline{\overline{K}}_{2,h}(z_2 - z_3)dz_2dz_3 = O(h^q) = o(1). \quad \square$$

For the next lemma, we recall that the goal is to prove that

$$\widehat{S}_{3,glb}(r) = O(h^2) + o_p(N^{-1/2}), \quad \widehat{S}_{4,glb}(r) = o_p(N^{-1/2}), \tag{2.13}$$

$$\widehat{S}_{4,glb}(r) = O(Nh^{q+s}) + O_p\Big(\sqrt{Nh^{2q+2s}}\Big) + o_p\Big(\sqrt{h^q}\Big), \tag{2.14}$$

$$\widehat{S}_{6,glb}(r) = O_p\Big(\sqrt{Nh^{2q+2s}}\Big) + o_p\Big(\sqrt{h^q}\Big). \tag{2.15}$$

Proof of Lemma [A.5]: For the proof of the first relation in (2.13), write $\widehat{S}_{3,glb}(r) = \|K\|_2^2N^{-1}\sum_{i=1}^N \beta_i$, where

$$\beta_i = \int_{\mathcal{H}} \mathrm{tr}\{C_1'\theta(Z_i)v(X_i)v(X_i)'D_1D_1'v(X_i)v(X_i)'\theta(Z_i)'C_1\}K_{2,h}(z - Z_i)dz.$$

In view of $\widehat{S}_{3,glb}(r) = \|K\|_2^2E\beta_i + O_p((N^{-1}E\beta_i^2)^{1/2})$, we examine next $E\beta_i$ and $E\beta_i^2$. With $F(z_i, z) = E(\mathrm{tr}\{C_1'\theta(Z_i)v(X_i)v(X_i)'D_1D_1'v(X_i)v(X_i)'\theta(Z_i)'C_1\}|Z_i = z_i)$, observe that $E\beta_i = \int_{\mathcal{H}} EF(Z_i, z)K_{2,h}(z - Z_i)dz$. Using Proposition 3.1, (i), below, $E\beta_i = \int_{\mathcal{H}}(F(z, z)p(z) + O(h^2))dz = O(h^2)$, since $F(z, z) = 0$ using $C_1(z)'\theta(z) = 0$. Similarly, $E\beta_i^2 = \int_{\mathcal{H}^2} EF(Z_i, z_1, z_2)K_{2,h}(z_1 - Z_i)K_{2,h}(z_2 - Z_i)dz_1dz_2$ with

$$F(z_i, z_1, z_2) = E\Big(\mathrm{tr}\{C_1(z_1)'\theta(Z_i)v(X_i)v(X_i)'D_1(z_1)D_1(z_1)'v(X_i)v(X_i)'\theta(Z_i)'C_1(z_1)\}\cdot$$

$$\cdot\,\mathrm{tr}\{C_1(z_2)'\theta(z_2)v(X_i)v(X_i)'D_1(z_2)D_1(z_2)'v(X_i)v(X_i)'\theta(z_2)'C_1(z_2)\}|Z_i = z_i\Big)$$

and, using Proposition 3.1, (ii) and (iii), $E\beta_i^2 = \int_{\mathcal{H}^2} F(z_1, z_1, z_2)\overline{K}_{2,h}(z_1 - z_2)dz_1dz_2 + o(1) = o(1)$, since $F(z_1, z_1, z_1) = 0$. The proof of the second relation in (2.13) is similar and easier since $E(U_i|X_i, Z_i) = 0$.

Write $\widehat{S}_{4,glb}(r) = 2\sum_{i<j} a_{ij}$ as a second order U-statistic with

$$a_{ij} = \frac{h^q}{N}\int_{\mathcal{H}} \mathrm{tr}\{C_1'\theta(Z_i)v(X_i)v(X_i)'D_1D_1'v(X_j)v(X_j)'\theta(Z_j)'C_1\}K_h(z - Z_i)K_h(z - Z_j)dz.$$

To show (2.14), we use Proposition 3.2 below. With $W_i = (Z_i, X_i)$, we need to examine Ea_{ij}, $E(E(a_{ij}|W_i)^2)$ and Ea_{ij}^2.

Observe that $Ea_{ij} = \frac{h^q}{N}\int_{\mathcal{H}} EF(Z_i, Z_j, z)K_h(z - Z_i)K_h(z - Z_j)dz$ with

$$F(z_i, z_j, z) = E(\mathrm{tr}\{C_1'\theta(Z_i)v(X_i)v(X_i)'D_1D_1'v(X_j)v(X_j)'\theta(Z_j)'C_1\}|Z_i = z_i, Z_j = z_j).$$

Using Proposition 3.1, (i), we obtain that

$$Ea_{ij} = \frac{h^q}{N}\left(\int_{\mathcal{H}} F(z, z, z)p(z)^2dz + O(h^s)\right) = O\left(\frac{h^{q+s}}{N}\right), \tag{2.16}$$

since $F(z, z, z) = 0$. Similarly, note that

$$E(a_{ij}|W_i) = \frac{h^q}{N}\int_{\mathcal{H}} \mathrm{tr}\{C_1'\theta(Z_i)v(X_i)v(X_i)'D_1D_1'\Big(Ev(X_j)v(X_j)'\theta(Z_j)'K_h(z - Z_j)\Big)C_1\}\cdot$$

$$\cdot K_h(z - Z_i)dz = \frac{h^q}{N}\int_{\mathcal{H}} \mathrm{tr}\{C_1'\theta(Z_i)v(X_i)v(X_i)'D_1D_1'O(h^s)C_1\}K_h(z - Z_i)dz$$

and, using Proposition 3.1, (ii) and (iii),

$$E(E(a_{ij}|W_i)^2) = \frac{h^{2q}}{N^2}\int_{\mathcal{H}^2} E\mathrm{tr}\{C_1(z_1)'\theta(Z_i)v(X_i)v(X_i)'D_1(z_1)D_1(z_1)'O(h^s)C_1(z_1)\}\cdot$$

$$\cdot\,\mathrm{tr}\{C_1(z_2)'\theta(Z_i)v(X_i)v(X_i)'D_1(z_2)D_1(z_2)'O(h^s)C_1(z_2)\}\cdot$$

$$\cdot\, K_h(z_1 - Z_i)K_h(z_2 - Z_i)dz_1dz_2 = O\left(\frac{h^{2q+2s}}{N^2}\right). \tag{2.17}$$

Furthermore,

$$Ea_{ij}^2 = \frac{h^{2q}}{N^2}\int_{\mathcal{H}^2} EF(Z_i, Z_j, z_1, z_2)K_h(z_1 - Z_i)K_h(z_2 - Z_i)K_h(z_1 - Z_j)K_h(z_2 - Z_j)dz_1dz_2,$$

where

$$F(z_i, z_j, z_1, z_2) = E\Big(\mathrm{tr}\{C_1(z_1)'\theta(Z_i)v(X_i)v(X_i)'D_1(z_1)D_1(z_1)'v(X_j)v(X_j)'\theta(Z_j)'C_1(z_1)\}\cdot$$

$$\cdot\,\mathrm{tr}\{C_1(z_2)'\theta(Z_i)v(X_i)v(X_i)'D_1(z_2)D_1(z_2)'v(X_j)v(X_j)'\theta(Z_j)'C_1(z_2)\}|Z_i = z_i, Z_j = z_j\Big).$$

Using Proposition 3.1, (ii) and (iii), we obtain that Ea_{ij}^2 behaves as

$$\frac{h^{2q}}{N^2}\int_{\mathcal{H}^2} F(z_1, z_1, z_1, z_2)p(z_1)^2(\overline{K}_h(z_1 - z_2))^2dz_1dz_2 = o\left(\frac{h^q}{N^2}\right). \tag{2.18}$$

The relation (2.14) now follows from (2.16)–(2.18) and Proposition 3.2. The relation (2.15) can be proved in a similar and easier way. □

3 Auxiliary results

The following localization properties of kernel functions were used many times above. The kernel K below is as in Assumption 1 of Section [2], and is of order s. We also use the notation $K_p(z)$, $\overline{K}(z)$, $\mathcal{H}_z$ and $\mathcal{H}$ of that section.

Proposition 3.1 *The following assertions hold:*

(i) For a function $g : \mathcal{H}_z \to \mathrm{I\!R}$ and a point z_0 in the interior of $\mathcal{H}_z$, we have

$$\int_{\mathcal{H}_z} g(z) K_h(z - z_0) dz = g(z_0) + r(h). \tag{3.1}$$

Here, (i.1) $r(h) = O(h^s)$ if g is s times continuously differentiable in a neighborhood of z_0; (i.2) $r(h) = O(h^s)$ uniformly in $\mathcal{H}$ if g is s times continuously differentiable on $\mathcal{H}_z$; (i.3) $r(h) = o(1)$ if g is continuous at z_0 and $s = 1$; (i.4) $r(h) = o(1)$ uniformly in $\mathcal{H}$ if g is continuous on $\mathcal{H}_z$ and $s = 1$.

(ii) For a function $g : \mathcal{H}_z \to \mathrm{I\!R}$, we have

$$\int_{\mathcal{H}_z} g(z) K_h(z - z_1) K_h(z - z_2) dz = g(z_1) \overline{K}_h(z_1 - z_2) + r(h). \tag{3.2}$$

Here, (ii.1) $r(h) = O(h\overline{K}_{1,h}(z_1 - z_2))$ uniformly over $z_1, z_2 \in \mathcal{H}$ if g is continuously differentiable on $\mathcal{H}_z$ and $s = 1$; (ii.2) $r(h) = o(\overline{K}_{1,h}(z_1 - z_2))$ uniformly over $z_1, z_2 \in \mathcal{H}$ if g is continuous on $\mathcal{H}_z$ and $s = 1$.

(iii) Let $\mathcal{H}_0$ denote either $\mathcal{H}$ or $\mathcal{H}_z$, and suppose $s = 1$. For a continuous function $g : \mathcal{H}_0 \times \mathcal{H}_0 \to \mathrm{I\!R}$, we have

$$\int_{\mathcal{H}_0^2} g(z_1, z_2) K_h(z_1 - z_2) dz_1 dz_2 = \int_{\mathcal{H}_0} g(z_1, z_1) dz_1 + o(1). \tag{3.3}$$

(iv) Let $\mathcal{H}_0$ denote either $\mathcal{H}$ or $\mathcal{H}_z$, and suppose $s = 1$. For a continuous function $g : \mathcal{H}_0^2 \times \mathcal{H} \to \mathrm{I\!R}$, we have

$$\int_{\mathcal{H}_0^2} \Big(\int_{\mathcal{H}} g(z_1, z_2, z_3) K_h(z_1 - z_2) K_h(z_1 - z_3) dz_1 \Big) dz_2 dz_3$$

$$= \int_{\mathcal{H}^2} g(z_2, z_2, z_3) \overline{K}_h(z_2 - z_3) dz_2 dz_3 + o(1). \tag{3.4}$$

PROOF: The statements $(i.1)$ and $(i.2)$ follow easily by using Taylor's expansions. The parts $(i.3)$ and $(i.4)$ are proved, for example, in Pagan and Ullah (1999), p. 362, even under weaker assumptions on the kernel K.

The statement $(ii.1)$ also follows easily by using Taylor's expansion. (Here, $\overline{K}_1$ is understood as the convolution of K_1.) For the part $(ii.2)$, observe that, for $z_1, z_2 \in \mathcal{H}$ and small enough h,

$$\begin{aligned}
&\Big|\int_{\mathcal{H}_z} g(z)K_h(z-z_1)K_h(z-z_2)dz - g(z_1)\overline{K}_h(z_1-z_2)\Big| \\
&= \Big|\int_{\mathcal{H}_z} (g(z)-g(z_1))K_h(z-z_1)K_h(z-z_2)dz\Big| \\
&\le \frac{C}{h^q}\int_{[-A,A]^q} |g(z_1+wh)-g(z_1)||K(w)|\Big|K\Big(w-\frac{z_2-z_1}{h}\Big)\Big|dw \\
&\le C \sup_{z_1\in\mathcal{H},|u|\le Ah} |g(z_1+u)-g(z_1)|\overline{K}_{1,h}(z_1-z_2),
\end{aligned}$$

where $\mathrm{supp}\{K\} \subset [-A,A]^q$. The supremum term above converges to zero as $h \to 0$ by the uniform continuity of g on compact supports.

To show (iii), suppose that $\mathcal{H}_0 = \mathcal{H}_z$ and, for simplicity, that $q = 1$, $\mathcal{H}_z = [0,1]$ and that the support of K is $[-1,1]$. Write

$$\int_{\mathcal{H}_z^2} g(z_1,z_2)K_h(z_1-z_2)dz_1dz_2$$

$$= \Big(\int_0^h\int_0^1 + \int_{1-h}^1\int_0^1 + \int_h^{1-h}\int_0^1\Big) g(z_1,z_2)K_h(z_1-z_2)dz_1dz_2 =: I_1 + I_2 + I_3.$$

As for part (i), one can easily show that

$$I_3 = \int_h^{1-h} g(z_1,z_1)dz_1 + o(1) = \int_0^1 g(z_1,z_1)dz_1 + o(1)$$

(note here that, for $z_1 \in [h, 1-h]$, the support of $K_h(z_1 - \cdot)$ is in $[0,1]$ and hence $\int_0^1 K_h(z_1 - z_2)dz_2 = 1$). Furthermore, for I_1, for example, note that

$$I_1 = h\int_0^1\int_0^{h^{-1}} g(hw_1,hw_2)K(w_1-w_2)dw_1dw_2 = O(h).$$

The part (iv) can be proved in a similar way. For example, with $q = 1$, $\mathcal{H}_0 = \mathcal{H} = [0,1]$ and K having the support $[-1,1]$, one would consider

$$\int_0^1\int_0^1\int_0^1 g(z_1,z_2,z_3)K_h(z_1-z_2)K_h(z_1-z_3)dz_1dz_2dz_3$$

$$= \int_h^{1-h} dz_2 \int_h^{1-h} dz_3 \left(\int_0^1 g(z_1, z_2, z_3) K_h(z_1 - z_2) K_h(z_1 - z_3) dz_1 \right) + R(h).$$

The first term behaves asymptotically as $\int_0^1 \int_0^1 g(z_2, z_2, z_3) \overline{K}_h(z_2 - z_3) dz_2 dz_3$, and the second term $R(h)$ is asymptotically negligible. □

Remark 3.1 When a function g depends on other variables w, for example, $g(z)$ replaced by $g(w, z)$ in (i), Proposition 3.1 obviously applies to such g for a fixed value of w. But the results of Proposition 3.1 also remain valid uniformly for w in a compact subset $\mathcal{H}_w$, as long as the assumptions involve obvious modifications. For example, $(i.2)$ would assume that $g(w, z)$ has its s order partial derivative with respect to z continuous on $\mathcal{H}_w \times \mathcal{H}_z$.

We also used the following result, borrowed from Fortuna (2008), Lemma C.1, p. 181. It concerns the limiting behavior of a second order U-statistic:

$$U_N = \sum_{1 \le i < j \le N} a_N(W_i, W_j), \tag{3.5}$$

where W_i, $i = 1, \dots, N$, are i.i.d. random vectors in $\mathbb{R}^d$ and $a_N : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a symmetric kernel (that is, $a_N(x, y) = a_N(y, x)$).

Proposition 3.2 *Let U_N be a second order U-statistic defined by (3.5). Then,*

$$U_N = \frac{N(N-1)}{2} E a_N(W_i, W_j) + O_p\left(\sqrt{N^3 E(E(a_N(W_i, W_j)|W_i)^2)} + \sqrt{N^2 E a_N(W_i, W_j)^2} \right). \tag{3.6}$$

The next four propositions were used in the proof of global rank tests.

Proposition 3.3 *Under Assumptions G4-5, the matrices C_0 and D_0 can be chosen analytic.*

Proof: By Assumptions G4 and G5, the matrix $\Sigma^{-1/2} g \psi^{-1/2}$ is analytic. By using the analytic Singular Value Decomposition (Bunse-Gerstner et al. (1991)), there are $m \times m$, $m \times n$ and $n \times n$ analytic matrices U, T and V, respectively, such that $\Sigma^{-1/2} g \psi^{-1/2} = UTV'$, where $T = \text{diag}(t_1, \dots, t_k)$ with $k = \min(m, n)$, $t_1 \ge \dots \ge t_k$ are the singular values, and orthogonal matrices U and V consist of the eigenvectors of $\Sigma^{-1/2} g \psi^{-1} g' \Sigma^{-1/2}$ and $\psi^{-1/2} g' \Sigma^{-1} g \psi^{-1/2}$, respectively. Now take $C_0 = \Sigma^{1/2} U$. Then, C_0 is analytic, satisfies $g \psi^{-1} g' \Sigma^{-1} C_0 = C_0 T^2$ and $C_0 \Sigma^{-1} C_0 = I_m$. The case of the matrix D_0 can be considered similarly. □

Proposition 3.4 *Under Assumptions 1-3, L4-5, and* $N^{1-2/u}h^q/\ln N \to \infty$, $Nh^{q+2s} \to 0$, *we have*

$$\sup_{z\in\mathcal{H}} \left|(\widehat{\psi}(z))^k - (\psi(z))^k\right| = O_p\left((Nh^q/\ln N)^{-1/2}\right), \quad k = -1, 1, \tag{3.7}$$

and

$$\sup_{z\in\mathcal{H}} |\widehat{\theta}(z) - \theta(z)| = O_p\left((Nh^q/\ln N)^{-1/2}\right). \tag{3.8}$$

Proof: To prove (3.7), we consider only the case $k = 1$. Under the assumptions of the proposition and by using Lemma B.1 in Newey (1994) (see also Lemma 1 in Fan and Zhang (1999)), we have

$$\sup_{z\in\mathcal{H}} |\widehat{\psi}(z) - E\widehat{\psi}(z)| = O_p\left((Nh^q/\ln N)^{-1/2}\right).$$

By using Proposition 3.1, (i), above and the assumptions,

$$\sup_{z\in\mathcal{H}} |E\widehat{\psi}(z) - \psi(z)| = O\left(h^s\right).$$

This implies (3.7) with $k = 1$. Relation (3.8) follows similarly by using (3.7). □

Proposition 3.5 *Under the assumptions of Proposition 3.4 above, we have*

$$\widehat{\Sigma} - \Sigma = O_p\left((Nh^q/\ln N)^{-1/2}\right).$$

Proof: Write

$$\begin{aligned}\widehat{\Sigma} &= \frac{1}{N\widehat{p}_{\mathcal{H}}}\sum_{i=1}^{N} U_iU_i'1_{\{Z_i\in\mathcal{H}\}} + \frac{1}{N\widehat{p}_{\mathcal{H}}}\sum_{i=1}^{N}(\theta(Z_i) - \widehat{\theta}(Z_i))v(X_i)v(X_i)'(\theta(Z_i) - \widehat{\theta}(Z_i))'1_{\{Z_i\in\mathcal{H}\}} \\ &\quad + \frac{1}{N\widehat{p}_{\mathcal{H}}}\sum_{i=1}^{N} U_iv(X_i)'(\theta(Z_i) - \widehat{\theta}(Z_i))'1_{\{Z_i\in\mathcal{H}\}} + \frac{1}{N\widehat{p}_{\mathcal{H}}}\sum_{i=1}^{N}(\theta(Z_i) - \widehat{\theta}(Z_i))v(X_i)U_i'1_{\{Z_i\in\mathcal{H}\}}.\end{aligned}$$

The first term on the right-hand side is $\Sigma + O_p(N^{-1/2})$. By using Proposition 3.4, the second term is $O_p((Nh^q/\ln N)^{-1})$ since $N^{-1}\sum_{i=1}^{N}|v(X_i)v(X_i)'| = O_p(1)$. Similarly, the third and fourth terms are $O_p((Nh^q/\ln N)^{-1/2})$ since $N^{-1}\sum_{i=1}^{N}|v(X_i)'U_i| = O_p(1)$. □

Proposition 3.6 *Let* $l(z) = \mathrm{rk}\{\Gamma(z)\}$. *Under the assumptions of Proposition 3.4 above, for* $i = 1, \ldots, m$,

$$\sup_{z\in\mathcal{H}} |\widehat{\lambda}_i(z) - \lambda_i(z)| = O_p\left((Nh^q/\ln N)^{-1/2}\right).$$

Proof: Observe by the Wielandt-Hoffman theorem (Golub and Van Loan (1996), Stewart and Sun (1990)) that

$$\sup_{z\in\mathcal{H}} \sum_{i=1}^{m} |\widehat{\lambda}_i(z) - \lambda_i(z)|^2$$
$$\leq \sup_{z\in\mathcal{H}} \left| \widehat{\Sigma}^{-1/2}\widehat{\Theta}(z)\widehat{\Psi}(z)\widehat{\Theta}(z)'\widehat{\Sigma}^{-1/2} - \Sigma^{-1/2}\Theta(z)\Psi(z)\Theta(z)'\Sigma^{-1/2} \right|^2,$$

which is $O_p((Nh^q/\ln N)^{-1})$ by Propositions 3.4 and 3.5. □

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Stephen G. Donald
Dept. of Economics, University of Texas at Austin
1 University Station C3100
Austin, TX 78712, USA
donald@eco.utexas.edu

Natércia Fortuna
Faculdade de Economia do Porto
Rua Dr. Roberto Frias
4200–464 Porto, Portugal
natercia.fortuna@gmail.com

Vladas Pipiras

Dept. of Statistics and Operations Research, UNC at Chapel Hill
CB#3260, Hanes Hall
Chapel Hill, NC 27599, USA
pipiras@email.unc.edu