Supplementary Material for "Stationary subspace analysis of nonstationary covariance processes: eigenstructure description and testing"

Raanju R. Sundararajan
Southern Methodist University
King Abdullah University of Science and Technology

Vladas Pipiras
University of North Carolina

Mohsen Pourahmadi
Texas A&M University

A Additional figures for Section 6

A.1 Figures for Section 6.1

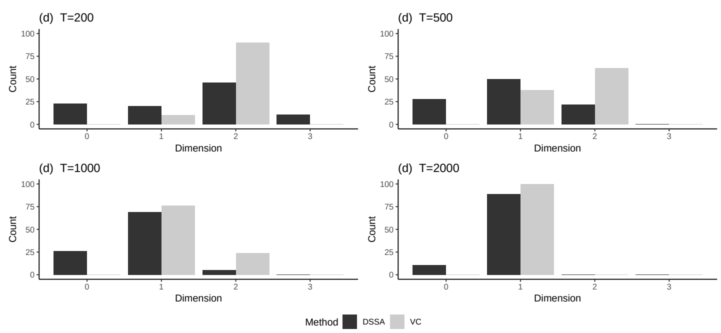

Figure 1: Model 1: Histograms of the estimates of d for the indicated sample sizes for the two competing methods: DSSA (Sundararajan and Pourahmadi (2018)) and VC (proposed method). The true value is $d = 1$.

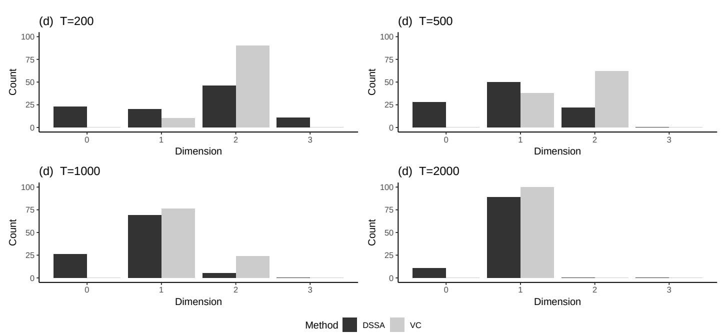

Figure 2: Model 2: Histograms of the estimates of d for the indicated sample sizes for the two competing methods: DSSA and VC (proposed method). The true value is $d = 2$.

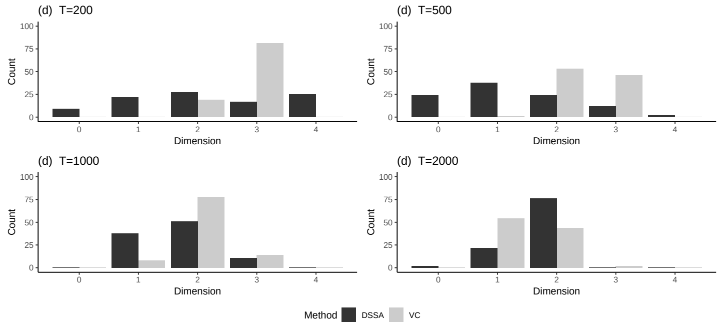

Figure 3: Model 3: Histograms of the estimates of d for the indicated sample sizes for the two competing methods: DSSA and VC (proposed method). The true value is $d = 2$.

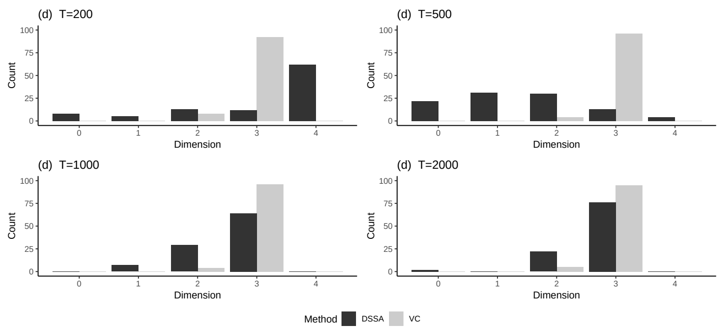

Figure 4: Model 4: Histograms of the estimates of d for the indicated sample sizes for the two competing methods: DSSA and VC (proposed method). The true value is $d = 3$. The value of ρ is set to 0.5.

A.2 Figures for Section 6.2

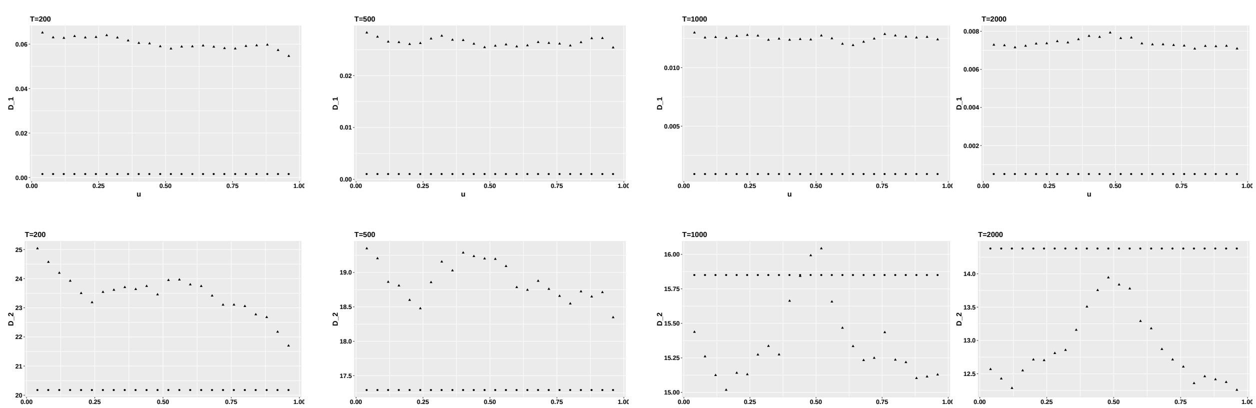

Figure 5: Model 2 - Top: Plot of $D_1(\widehat{B}_1(u))$ against u for the competing methods DSSA and VC and several sample sizes. VC (avg.) in triangles in squares and DSSA in solid circles. Bottom: Analogous plot but with measure $D_2(\widehat{B}_1(u))$ against u.

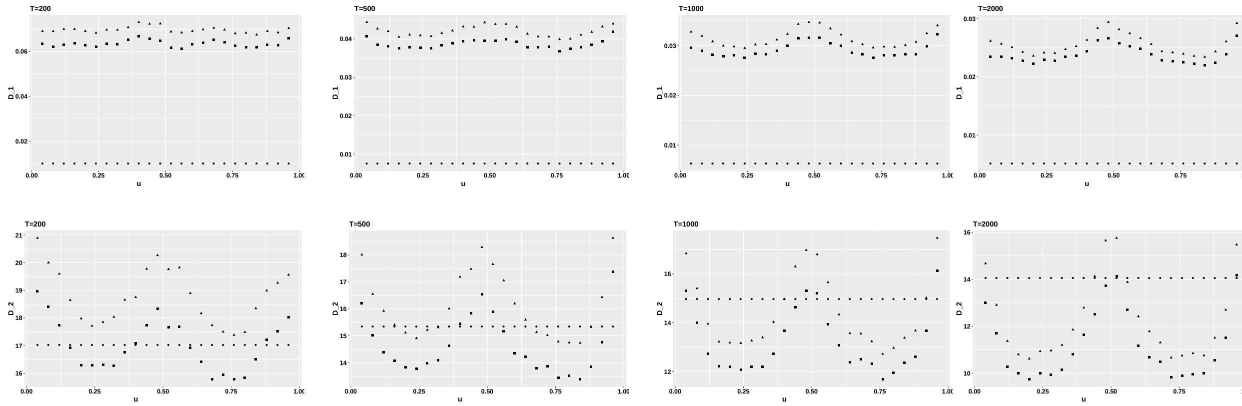

Figure 6: Model 3 - Top: Plot of $D_1(\widehat{B}_1(u))$ against u for the competing methods DSSA and VC and several sample sizes. VC (avg.) in triangles, VC (min.) in squares and DSSA in solid circles. Bottom: Analogous plot but with measure $D_2(\widehat{B}_1(u))$ against u.

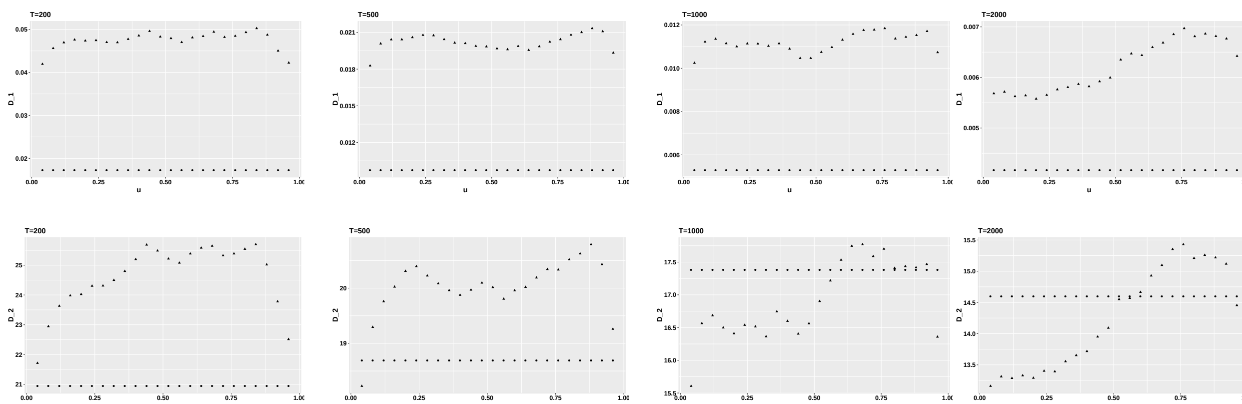

Figure 7: Model 4 - Top: Plot of $D_1(\widehat{B}_1(u))$ against u for the competing methods DSSA and VC and several sample sizes. VC (avg.) in triangles, VC (min.) in squares and DSSA in solid circles. Bottom: Analogous plot but with measure $D_2(\widehat{B}_1(u))$ against u.

B Additional figures and tables for Section 7

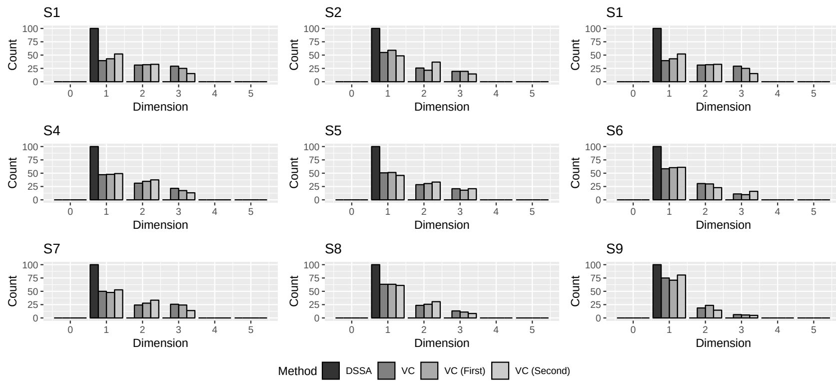

Figure 8: Histogram of the dimension estimates d by the two competing methods based on the 144 trials.

d		S1	S2	S3	S4	S5	S6	S7	S8	S9	Avg
1	DSSA	51.22	54.86	56.25	58.33	49.03	45.13	54.16	51.11	51.38	52.38
	VC	50	49.31	50.69	46.52	52.08	54.16	50.69	46.15	54.86	50.49
2	DSSA	58.37	57.63	54.86	61.11	54.16	52.08	52.77	56.45	52.78	55.57
	VC	53.14	59.72	48.61	57.63	55.56	52.77	54.17	45.05	50.69	53.10
3	DSSA	60.48	58.33	59.02	55.56	59.33	56.20	62.50	64.39	56.94	59.19
	VC	60.13	61.11	47.22	56.94	57.63	55.56	61.81	54.54	54.16	56.57
4	DSSA	60.17	62.50	56.25	66.67	62.50	55.56	65.27	66.28	55.56	61.19
	VC	58.04	60.41	65.97	66.67	64.58	57.63	66.67	59.44	56.94	61.82

Table 1: Out-of-sample classification accuracy (in %) for the 9 subjects S1-S9 for the two indicated methods.

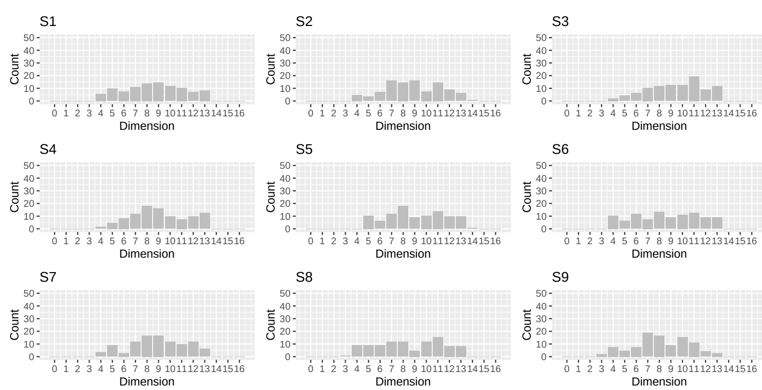

Figure 9: $p = 22$. Histogram of the stationary subspace dimension estimates based for VC method based on the 144 trials.

d	Method	S1	S2	S3	S4	S5	S6	S7	S8	S9	Avg
7	VC	74.12	66.67	70.83	73.51	73.61	70.83	74.30	68.53	69.44	71.32
	DSSA	69.67	67.57	70.00	64.52	60.48	63.70	66.67	68.30	68.55	66.60
9	VC	80.41	75	72.22	77.08	81.25	74.30	77.08	71.32	77.08	76.19
	DSSA	71.52	69.96	77.90	77.48	65.50	70.55	71.41	70.38	69.80	71.61
11	VC	87.40	84.72	74.30	81.94	88.19	81.25	82.63	81.25	81.25	82.55
	DSSA	75.78	69.97	72.92	75.96	69.58	71.95	68.34	71.33	73.31	72.12
13	VC	89.50	90.97	85.41	88.19	89.58	88.19	90.27	84.72	84.72	87.95
	DSSA	79.86	70.98	80.56	78.24	70.83	77.62	73.55	78.38	74.72	76.08

Table 2: Out-of-sample classification accuracy (in %) for the 9 subjects S1-S9 corresponding to $d = 7, 9, 11$, and 13 for the VC and DSSA methods with $p = 22$.

C Consistent estimator of fourth moment

The asymptotic results in Propositions 4.1 and 4.2 involve the quantity $\mu_4 = \mathbb{E}(Y_{i,t}^2 - 1)^2 = \mathbb{E}Y_{i,t}^4 - 1$, that is, the fourth moment of the variables $Y_{i,t}$ in the VC model (1.2). One natural estimator of μ_4 is to set

$$\widehat{\mu}_4 = \frac{1}{Tp} \sum_{i=1}^{p} \sum_{t=1}^{T} \Big((\widehat{A}(\frac{t}{T})^{-1} X_t)_i^4 - 1 \Big). \tag{C.1}$$

But analyzing this estimator would require being able to control $\widehat{A}(u)$ uniformly across u. This is certainly possible, and is essentially done in connection to global tests, but would require more stringent assumptions than those used for local tests in Appendix A.1 of the paper. Instead of (C.1), a consistent estimator of μ_4 can be constructed under the assumptions of Appendix A.1 of the paper based on the following argument.

To get to the fourth moment of $Y_{i,t}$, we shall consider $(\mathrm{tr}\{X_t X_t'\})^2$. Note that

$$\begin{aligned}\mathrm{tr}\{X_t X_t'\} &= \mathrm{tr}\{A(\frac{t}{T}) Y_t Y_t' A(\frac{t}{T})'\} = \mathrm{tr}\{A(\frac{t}{T})' A(\frac{t}{T}) Y_t Y_t'\} = \mathrm{tr}\{A^2(\frac{t}{T}) Y_t Y_t'\} \\ &= \mathrm{tr}\{A^2(\frac{t}{T})(Y_t Y_t' - I_p)\} + \mathrm{tr}\{A^2(\frac{t}{T})\},\end{aligned}$$

where we used the symmetry of $A(u)$ to write $A(\cdot)'A(\cdot) = A(\cdot)A(\cdot)' = A^2(\cdot)$. Then,

$$\begin{aligned}
\frac{1}{T}\sum_{t=1}^{T}(\mathrm{tr}\{X_tX_t'\})^2 &= \frac{1}{T}\sum_{t=1}^{T}\Big(\mathrm{tr}\{A^2(\frac{t}{T})(Y_tY_t' - I_p)\}\Big)^2 \\
&\quad + \frac{2}{T}\sum_{t=1}^{T}\mathrm{tr}\{A^2(\frac{t}{T})(Y_tY_t' - I_p)\}\mathrm{tr}\{A^2(\frac{t}{T})\} \\
&\quad + \frac{1}{T}\sum_{t=1}^{T}\Big(\mathrm{tr}\{A^2(\frac{t}{T})\}\Big)^2 =: R_1 + R_2 + R_3. \qquad (C.2)
\end{aligned}$$

Under the assumptions of Appendix A.1 of the paper, note that $R_2 \to 0$ a.s. Indeed, this follows from the following general argument that will be used on several occasions below. After expanding the traces, an entry in R_2 can be expressed as

$$\frac{1}{T}\sum_{t=1}^{T} b(\frac{t}{T})Z_t, \qquad (C.3)$$

where $b(\cdot)$ is continuously differentiable and Z_t's are i.i.d. with zero mean. By the summation by parts formula,

$$\frac{1}{T}\sum_{t=1}^{T} b(\frac{t}{T})Z_t = \frac{1}{T}\sum_{t=1}^{T-1}\Big(\sum_{s=1}^{t} Z_s\Big)\Big(b(\frac{t}{T}) - b(\frac{t+1}{T})\Big) + \frac{b(1)}{T}\sum_{t=1}^{T} Z_t - \frac{b(1/T)}{T}\sum_{t=1}^{T-1} Z_t.$$

All three last terms converge to 0 a.s. by the law of large numbers. For the first term, in particular, this follows from bounding it by

$$C\sum_{t=1}^{T}\Big|\sum_{s=1}^{t} Z_s\Big|\frac{1}{T^2} = C\sum_{t=1}^{T}\Big|\frac{1}{t}\sum_{s=1}^{t} Z_s\Big|\frac{t}{T^2}$$

and noting that $\frac{1}{t}\sum_{s=1}^{t} Z_s \to 0$ a.s. as $t \to \infty$.

Under the assumptions of Appendix A.1 of the paper, $R_3 \to \int_0^1 (\mathrm{tr}\{A^2(u)\})^2 du$. For the term R_1, note that

$$R_1 = \frac{1}{T}\sum_{t=1}^{T}\Big(\sum_{i,j=1}^{p} A_{ij}^2(\frac{t}{T})(Y_{i,t}Y_{j,t} - \delta_{ij})\Big)^2 =: \frac{1}{T}\sum_{t=1}^{T} R_{1,t},$$

where $\delta_{ij} = 1$ if $i = j$, and $= 0$ otherwise. By separating the sum $R_{1,t}$ into that over $i = j$ and that over $i < j$, and taking the square, we can further write

$$R_{1,t} = R_{1,1,t} + R_{1,2,t} + R_{1,3,t} + R_{1,4,t} + R_{1,5,t},$$

where

$$
\begin{aligned}
R_{1,1,t} &= \sum_{i=1}^{p}(A_{ii}^2(\frac{t}{T}))^2(Y_{i,t}^2-1)^2, \\
R_{1,2,t} &= \sum_{i\neq i'} A_{ii}^2(\frac{t}{T})A_{i'i'}^2(\frac{t}{T})(Y_{i,t}^2-1)(Y_{i',t}^2-1), \\
R_{1,3,t} &= 4\sum_{i'=1}^{p} A_{i'i'}^2(\frac{t}{T})(Y_{i',t}^2-1)\sum_{i<j} A_{ij}^2(\frac{t}{T})Y_{i,t}Y_{j,t}, \\
R_{1,4,t} &= 4\sum_{i<j}(A_{ij}^2(\frac{t}{T}))^2 Y_{i,t}^2 Y_{j,t}^2, \\
R_{1,4,t} &= 4\sum_{i<j}\sum_{i'<j'} 1_{\{i\neq i' \text{ or } j\neq j'\}} A_{ij}^2(\frac{t}{T})A_{i'j'}^2(\frac{t}{T})Y_{i,t}Y_{j,t}Y_{i',t}Y_{j',t}.
\end{aligned}
$$

By the same reasoning following (C.3), for $k=2,3,5$

$$
\frac{1}{T}\sum_{t=1}^{T} R_{1,k,t} \to 0 \quad \text{a.s.}
$$

and

$$
\frac{1}{T}\sum_{t=1}^{T} R_{1,1,t} \to \mu_4 \int_0^1 \sum_{i=1}^{p}(A_{ii}^2(u))^2 du, \quad \frac{1}{T}\sum_{t=1}^{T} R_{1,4,t} \to 4\int_0^1 \sum_{i<j}(A_{ij}^2(u))^2 du \quad \text{a.s.}
$$

By gathering the above observations, it follows from (C.2) that, almost surely,

$$
\frac{1}{T}\sum_{t=1}^{T}(\mathrm{tr}\{X_t X_t'\})^2 \to \mu_4 \int_0^1 \sum_{i=1}^{p}(A_{ii}^2(u))^2 du + 4\int_0^1 \sum_{i<j}(A_{ij}^2(u))^2 du =: \mu_4 I_1 + 4I_2. \tag{C.4}
$$

We indicate how the integrals I_1 and I_2 can be estimated consistently, from which a consistent estimator of μ_4 will follow.

The integrals I_1 and I_2 can be estimated through the following weighted U-statistics:

$$
\begin{aligned}
\widehat{I}_1 &= \frac{1}{T(T-1)}\sum_{t_1\neq t_2}\sum_{i=1}^{p}(X_{t_1}X_{t_1}')_{ii}(X_{t_2}X_{t_2}')_{ii}K_h(\frac{t_1}{T}-\frac{t_2}{T}), \\
\widehat{I}_2 &= \frac{1}{T(T-1)}\sum_{t_1\neq t_2}\sum_{i<j}(X_{t_1}X_{t_1}')_{ij}(X_{t_2}X_{t_2}')_{ij}K_h(\frac{t_1}{T}-\frac{t_2}{T}).
\end{aligned}
$$

The convergence of $\widehat{I}_1$ and $\widehat{I}_2$ to I_1 and I_2, respectively, can be attained by using Proposition A.3 of the paper, which also yields convergence rates.

References

Sundararajan, R. R. and M. Pourahmadi (2018). Stationary subspace analysis of nonstationary processes. *Journal of Time Series Analysis 39*(3), 338–355.